Quantum States

Early 20th Century physics

- classical mechanics viewed matter as composed of particles, and light as composed of continuous electromagnetic waves
- **diffraction experiment:** beam of subatomic particles hitting a crystal diffract in a wave-like pattern
  - de Broglie wavelength associated with matter
- **photoelectric effect:** an atom hit by a beam of light may absorb it, causing electrons transition to a higher energy orbital
  - absorbed energy may be emitted as light causing electrons to transition back to the original orbital
  - light-matter transactions always occur via discrete packets of energy, i.e. photons
- further experimental evidence: old duality particle-wave theory needed to be replaced by a theory in which **both** matter and light can exhibit both particle- and wave-like behaviour.
- **Young's double slit experiment:** shine light at a boundary with 2 very close slits, between the light source and an observing wall
  - pattern of light on the wall varies between light and dark as a result of interference between light
  - with one slit closed, no interference pattern is observed
  - remarkable results:
    * double-slit experiment can be performed with a single photon: if there is a single photon, why would there be any interference pattern?
    * can also be performed with electrons, protons, atomic nuclei, bucky balls, all of which exhibit interference behaviour
- conclusion: rigid distinction between waves and particles as a means of describing the physical world is untenable at the quantum level

Quantum States

Particle on a line

- consider a subatomic particle on a line that may only be found at one of several equally spaced points $\{x_0, ..., x_{n-1}\}$ separated by distance δx

- describe the current state of the particle as a complex vector $[c_0, ..., c_{n-1}]^T$
- denote the particle being at point i as $|x_i\rangle$ (a **ket**)
- each basic state has an associated column vector $|x_i\rangle \to \delta_{ij} \in \mathbb{C}^n$
- note these vectors form the canonical basis of $\mathbb{C}^n$
- in quantum physics, the particle can be in a fuzzy blending of states: all vectors in $\mathbb{C}^n$ represent a legitimate physical state
- **superposition:** an arbitrary state $|\psi\rangle$ is then a linear combination of the basic states $|x_i\rangle, ..., |x_{n-1}\rangle$ with **complex amplitudes** $c_0, ..., c_{n-1}$
  - represents particle being simultaneously in all locations, a blending of all $|x_i\rangle$

$$|\psi\rangle = c_0 |x_0\rangle + ... + c_{n-1} |x_{n-1}\rangle$$

- every state can therefore be represented as an element of $\mathbb{C}^n$ as

$$|\psi\rangle \to [c_0, ..., c_{n-1}]^T$$

- probability that, after observing the particle, we will detect it at point x_i:

$$p(x_i) = \frac{|c_i|^2}{||\psi\rangle|^2} = \frac{|c_i|^2}{\sum_j |c_j|^2}$$

- clearly $p(x_i) \in \mathbb{R}$ and $0 \leq p(x_i) \leq 1$
- when $|\psi\rangle$ is observed, it will be found in one of the basic states
- kets can be added: $|\psi\rangle + |\psi\rangle' = [c_0 + c_0', ..., c_{n-1} + c_{n-1}']^T$
- a ket $|\psi\rangle$ and its scalar multiples $c|\psi\rangle$ (for some $c \in \mathbb{C}$) describe the same physical state
- the length of $|\psi\rangle$ doesn't matter as far as physics goes
- it then makes sense to work with a **normalised ket** with length 1:

$$\frac{|\psi\rangle}{||\psi\rangle|}$$

- for a normalised ket, we have $p(x_i) = |c_i|^2$

Spin

- property of subatomic particles which is the prototypical way to implement qubits
- **Stern-Gerlach experiment:** electron in presence of magnetic field observed to behave as if it were a charged spinning top, by acting as a magnet and trying to align itself with the magnetic field

  - experiment: shoot beam of electrons through a magnetic field oriented in a certain direction
  - beam is split into 2 streams with opposite spin

- differences to classical spinning top:
  - electron doesn't have internal structure: quantum property with no classical analog
  - all electrons can be found in 1 of 2 locations, not distributed between (spin can be clockwise/anticlockwise)

- for each direction in space, there are only 2 spin states, spin **up** $|\uparrow\rangle$ and **down** $|\downarrow\rangle$
- arbitrary state is then a superposition of up and down:

$$|\psi\rangle = c_0 |\uparrow\rangle + c_1 \downarrow$$

- inner product: modifies vector space into a space with *geometry*, adding angles, orthogonality, distance
- inner product of state space allows computation of **transition amplitudes**, which you can use to determine the likelihood the state of the system before a specific measurement will change to another state after measurement has occurred
- consider two normalised states $|\psi\rangle$, $|\psi'\rangle$
- let the start state be $|\psi\rangle$, and the end state a row vector with complex conjugate coordinates of $|\psi'\rangle$
- define the **bra**_ $\langle\psi'| = |\psi\rangle^\dagger = [\overline{c'_0}, ..., \overline{c'_{n-1}}]$
- the transition amplitude is then the inner product **bra-ket**

$$\langle\psi'|\psi\rangle = [\overline{c'_0}, ..., \overline{c'_{n-1}}] \begin{bmatrix} c_0 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

- represent the start state, end state, and amplitude of going between these states as:

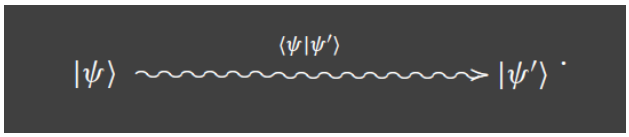

Figure 1: Transition Amplitude Diagram

- bra-ket approach shifts focus from states to state transitions
- the transition amplitude between two states is zero when two states are orthogonal: orthogonal states are mutually exclusive alternatives

- e.g. an electron can be in arbitrary superposition of spin up and down, but after measurement in the z-direction, it will always be *either* up or down, not *both* up and down
  - if the electron was already in the up state before the z-direction measurement, it will never transition to a down state as a result of that measurement
- every complete measurement of a quantum system has an associated orthonormal basis of all possible outcomes
- with $|\psi\rangle$ in the basis $\{|b_0\rangle, ..., |b_{n-1}\rangle\}$, i.e.

$$|\psi\rangle = \sum_{i=0}^{n-1} b_i |b_i\rangle$$

- each $|b_i|^2$ is the probability of ending up in state $|b_i\rangle$ after a measurement has been made

Summary

- we can associate a vector space with a quantum system, with its dimension reflecting the number of basic states of the system
- states can be superposed by adding their representing vectors
- a state is left unchanged if its representing vector is multiplied by a complex scalar
- the state space has a geometry given by its inner product: this has a physical meaning, namely the likelihood of a given state to transition to another one after measurement
- orthogonal states are mutually exclusive

Observables

- physical quantities only make sense with respect to a quantifiable observation
- a physical system can be specified by a pair: *(state space, observables)*
  - **state space:** set of all states the system may occupy
  - **observables:** set of physical quantities able to be observed in each state of the state space
- each observable can be considered a question we can pose to the system: if the system is in a particular state $|\psi\rangle$ what values can we observe?
- **Postulate:** each physical observable has a corresponding hermitian operator
  - *reminder:* Hermitian means $A^\dagger = A$
  - an observable is a linear operator: it maps states to states
  - the application of an observable Ω to a state vector $|\psi\rangle$ is the resulting state $\Omega |\psi\rangle$

  - in general $\Omega |\psi\rangle$ is not a scalar multiple of $|\psi\rangle$; they do not represent the same state, i.e. Ω has modified the state of the system

- **Postulate:** let Ω be a hermitian operator associated with a physical observable. Then the eigenvalues of Ω are the only possible values the observable can take as a result of measuring it on any given state. The eigenvectors of Ω form a basis for the state space.
- so observables can be considered legitimate questions we can pose to quantum systems. The question may be answered with the eigenvalues of the observable

Position

- specific question: "where can the particle be found?"
- what's the corresponding hermitian operator, P, for position?
  - how does it operate on basic states e.g. $|x_i\rangle$? $P(|\psi\rangle) = P(|x_i\rangle) = x_i |\psi\rangle$: P acts as multiplication by position
  - the basic states form a basis, so for an arbitrary stat: $P(\sum c_i |x_i\rangle) = \sum x_i c_i |x_i\rangle$
- as a matrix: this is the diagonal matrix whose entries are the x_i coordinates
- note:
  - P is trivially hermitian
  - all diagonal elements are real
  - eigenvalues are x_i values
  - normalised eigenvectors are the basic state vectors

Momentum

- specific question: "what is the particle's momentum?"
- represented by operator M, proportional to the rate of change of the state vector across space

$$M(|\psi\rangle) = -i\hbar \frac{\partial |\psi\rangle}{\partial x}$$

Spin

- specific question: "for a given direction in space, in which direction is the particle spinning?"
  - e.g. up/down in z direction? left/right in x direction? in/out in y direction?
- spin operators:

$$S_z = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, S_y = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, S_x = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

- each spin operator has a corresponding orthonormal basis:
  - $S_z : \{|\uparrow\rangle, |\downarrow\rangle\}$, up and down
  - $S_y : \{|\leftarrow\rangle, |\rightarrow\rangle\}$, left and right
  - $S_x : \{|\swarrow\rangle, |\nearrow\rangle\}$, in and out

Manipulating Observables

- in physics we frequently add, multiply quantities to produce other meaningful quantities: momentum as mass*velocity, …
- to what extent can quantum observables be manipulated to obtain other observables?
- ✓ multiplication by a real scalar, $c \in \mathbb{R}, c\Omega$
  - Multiplying a hermitian matrix by a real scalar produces a hermitian matrix
- $\times$ multiplication by a complex scalar: the result may not be hermitian
- ✓ addition of two hermitian matrices $\Omega_1 + \Omega_2$
- set of hermitian matrices of fixed dimension forms a $\mathbb{R}$ vector space (but not a $\mathbb{C}$ one)
- products? e.g. $\Omega_1 \cdot \Omega_2$. Issues:
  - the order in which operators are applied to state vectors matters in general, as matrix multiplication is not generally commutative
  - the product of 2 hermitian operators is not guaranteed to be hermitian
- what does it take for the product of 2 hermitian operators to be hermitian?
  - recall $\langle H \cdot V, W\rangle = \langle V, H \cdot W\rangle$ for hermitian H. Accordingly for hermitian Ω_1, Ω_2:

$$\langle \Omega_1 \cdot \Omega_2\phi, \psi\rangle = \langle \Omega_2\phi, \Omega_1\psi\rangle = \langle \phi, \Omega_2 \cdot \Omega_1\psi\rangle$$

- for $\Omega_1 \cdot \Omega_2$ to be hermitian, we need:

$$\langle \Omega_1 \cdot \Omega_2\phi, \psi\rangle = \langle \phi, \Omega_1 \cdot \Omega_2\psi\rangle$$

- which implies we need $\Omega_1 \cdot \Omega_2 = \Omega_2 \cdot \Omega_1$
- we define the **commutator** operator as:

$$[\Omega_1, \Omega_2] = \Omega_1 \cdot \Omega_2 - \Omega_2 \cdot \Omega_1$$

- if $[\Omega_1, \Omega_2] = 0$, then the product $\Omega_1 \cdot \Omega_2 = \Omega_2 \cdot \Omega_1$ is hermitian
- e.g. $[S_x, S_y] = 2iS_z$, i.e. the spin operators do not commute
- note that the product of a hermitian operator with itself always commutes, as does the exponent operation. Therefore for a single hermitian Ω, we get the entire algebra of polynomials over Ω, i.e. all operators of the following form commute with one another:

$$\Omega' = \alpha_0 + \alpha_1\Omega + \alpha_2\Omega^2 + ... + \alpha_{n-1}\Omega^{n-1}$$

- consequently if the commutator of 2 hermitian operators is 0 (i.e. the operators commute), you are able to to assign their product as the mathematical equivalent of the physical product of their associated observables
- if the commutator is non-zero, we get Heisenberg's uncertainty principle

Expected Value

- hermitian operators are those which behave well with respect to the inner product: $\langle \Omega\phi, \psi\rangle = \langle \phi, \Omega\psi\rangle$ for each pair $|\psi\rangle, |\psi\rangle$
  - this means $\langle \Omega\psi, \psi\rangle \in \mathbb{R}$ for each ψ, denoted $\langle \Omega \rangle_\psi$
  - subscript denotes dependence on state vector
- **Postulate:** $\langle \Omega \rangle_\psi$ is the **expected value** of observing Ω repeatedly on the same state ψ
  - let $\lambda_1, ..., \lambda_{n-1}$ be the eigenvalues of Ω
  - prepare the system so that it is in state $|\psi\rangle$ and let us observe the value of Ω: this will yield one of the λ_i
  - repeat this n times, such that each λ_i has been seen p_i times
  - now compute the estimated expected value of Ω as $\frac{1}{n}\sum \lambda_i p_{i_i}$
  - if n is sufficiently large, this will be very close to $\langle \Omega\psi, \psi\rangle$

Variance

- the **variance** will indicate the spread of distribution around expected value
- introduce the hermitian operator

$$\Delta_\psi(\Omega) = \Omega - \langle \Omega \rangle_\psi I$$

- this operates on a generic vector $|\phi\rangle$ as:

$$\Delta_\psi(\Omega)\,|\phi\rangle = \Omega(|\phi\rangle) - (\langle \Omega \rangle_\psi)\,|\phi\rangle$$

- i.e. $\Delta_\psi(\Omega)$ substracts the mean from the result of Ω
- variance of Ω at $|\psi\rangle$ is then the expectation value of $\Delta_\psi(\Omega)$ squared:

$$Var_\psi(\Omega) = \langle(\Delta_\psi(\Omega)) \cdot (\Delta_\psi(\Omega))\rangle_\psi$$

- note this is not too far from $Var(X) = E((X-\mu)^2)$
- the variance of the same hermitian varies from state to state: on an eigenvector of the operator, the variance is 0, and the expected value is the corresponding eigenvalue: the observable is sharp on its eigenvectors; there is no ambiguity of outcome

Heisenberg's Uncertainty Principle

- consider observables represented by hermitians Ω_1, Ω_2 and a given state $|\psi\rangle$
- compute $Var_\psi(\Omega_1), Var_\psi(\Omega_2)$. Do they relate, and if so, how?
- i.e. given 2 observables we would hope to simultaneously minimise each variance such that the outcome was sharp for both
- if the variances were not correlated, you would expect a sharp measure of each observable on a convenient state
- however the variances are correlated
- **Theorem: Heisenberg's uncertainty principle** the product of the variances of 2 arbitrary hermitian operators on a given state is always greater than or equal to one quarter of the square of the expected value of their commutator:

$$Var_\psi(\Omega_1) \cdot Var_\psi(\Omega_2) \geq \frac{1}{4} |\langle[\Omega_1, \Omega_2]\rangle_\psi^2$$

- so the commutator measures how good a simultaneous measure of 2 observables can possibly be
- if the commutator happens to be 0, there is no fundamental limit to the accuracy
- however there are plenty of operators that do not commute e.g. directional spin operators
- position-momentum also do not commute. The expression of $|\psi\rangle$ with respect to the eigenbasis of each observable paints markedly different stories
  - $|\psi\rangle$ can be expressed in the momentum eigenbasis, which treats $|\psi\rangle$ like a wave, decomposing it into sinusoids
  - $|\psi\rangle$ expressed in the position eigenbasis is made of **Dirac deltas**, peaks zero everywhere except at a point, i.e. decomposed into a weighted sum of peaks

Summary

- observables are represented by hermitian matrices
- the result of an observation is always an eigenvalue of the hermitian
- $\langle\psi|\Omega|\psi\rangle$ represents the expected value of observing Ω on $|\psi\rangle$
- observables do not commute (in general): this means the order of observation matters, and that there is a fundamental limit on our ability to simultaneously measure their values

Measurement

- **measurement:** act of carrying out an observation on a physical system
  - observable corresponds to specific question posed
  - measuring is the process of asking a specific question and receiving a definite answer
- classical physics made the false implicit assumptions that
  - the act of measuring does not change the state of the system
  - the result of a measurement on a well-defined state is predictable: if a state is known with certainty, the value of the observable on that state can be anticipated
- these assumptions are wrong:
  - systems are perturbed as a result of measurement
  - only the probability of observing specific values can be calculated: measurement is inherently nondeterministic
- so far we know that as the result of an observation, an observable can only assume one of its eigenvalues
- how frequently will we see a given eigenvalue λ? What happens to the state vector if λ is observed?
- **Postulate:** let Ω be an observable, and $|\psi\rangle$ be a state. If the result of measuring Ω is the eigenvalue λ, the state after measurement will always be the eigenvector $|e\rangle$corresponding to λ.
  - we say that the system has **collapsed** from $|\psi\rangle$ to $|e\rangle$
- what is the probability that a (normalised) start state $|\psi\rangle$ will transition to a specific eigenvector $|e\rangle$?
  - this is given by the square of the inner product of the states, $\langle e|\psi\rangle^2$
  - this has the geometrical meaning of the **projection** of $|\psi\rangle$ along $|e\rangle$

Meaning of expected value

- remember the normalised eigenvectors of Ω form an orthogonal basis of the state space, so we can express $|\psi\rangle$ as a linear combination w.r.t. this basis: $|\psi\rangle = \sum c_i |e_i\rangle$
- compute the mean

$$\langle\Omega\rangle_\psi = \langle\Omega\psi, \psi\rangle = \sum |c_i|^2 \lambda_i$$

- this is exactly the mean value of the probability distribution $(\lambda_0, p_0), ..., (\lambda_{n-1}, p_{n-1})$
  - p_i: square amplitude of collapse into the corresponding eigenvector
- after measuring an observable, the system transitions to the corresponding eigenvector. If you ask the same question again, you will get the same answer
  - what if you change the question?

Order Matters

- consider making successive measurements for different observables
- each observable has a different set of eigenvectors to which the system will collapse
- the answer will depend on the order in which questions are posed
- e.g. polarising sheet and a beam of light
  - light can be **polarised**, where the wave only vibrates along a specific plane orthogonal to propogation (as opposed to all possible planes)
  - polarising sheet placed within the beam of light
    - measures polarisation of light in orthogonal basis corresponding to direction of the sheet
    - filters out photons that collapsed to one of the elements of the basis
  - adding a 2nd sheet
    - oriented in same direction: no difference whatsoever. Asking the same question repeatedly
    - rotated by 90°: no light passes through. The light that was not filtered by the first sheet is now guaranteed to be filtered by the second.
  - adding a 3rd sheet before/after 1st/2nd
    - no effect. No light permitted before, and none allowed through the additional sheet
  - placing a 3rd sheet in the middle, at 45°:
    - light passes through all three sheets:

- left sheet: measures all light relative to up-down basis
- light in vertical polarisation state that goes through is then in a superposition with respect to the basis of the diagonal sheet
- the middle sheet then collapses half, filters some, and passes some through
- the light passed through is again in a superposition with respect to the 3rd sheet, so some light again passes through
- note: with 50% filtering by each sheet, only 1/8 of the original light passes through

Summary

- the end state of a measurement of an observable is always one of its eigenvectors
- the probability for an initial state to collapse into an eigenvector of the observable is given by the length squared of the projection
- when measuring several observables sequentially, the order of measurement matters

Dynamics

- so far we have considered static quantum systems, so we need **quantum dynamics** to examine how quantum systems evolve over time

Unitary Transformations

- **Postulate:** the evolution of a quantum system (that is not a measurement) is given by a **unitary operator or transformation**
  - if U is a unitary matrix representing a unary operator, and $|\psi(t)\rangle$ represents a state of the system at time t, then:

$$|\psi(t+1)\rangle = U\,|\psi(t)\rangle$$

- properties of unitary transformations
  - **closed under composition:** the product of 2 arbitrary unitary matrices is unitary
  - **closed under inverse:** the inverse of a unitary matrix is unitary
  - **multiplicative identity:** the identity operator is trivially unitary
- the set of transformations constitutes a **group of transformations** with respect to composition

System evolution

- assume we have a rule $\mathfrak{U}$ that associates with each instant of time t_i a unitary matrix $\mathfrak{U}[t_i]$
- initial state vector $|\psi\rangle$
- you can then apply each $\mathfrak{U}[t_i]$ to form a sequence of state vectors

$$\mathfrak{U}[t_0]\,|\psi\rangle\,, ..., \mathfrak{U}[t_{n-1}]...\mathfrak{U}[t_0]\,|\psi\rangle$$

- this sequence is called an **orbit** of $|\psi\rangle$ under the action of $\mathfrak{U}[t_i]$ at time click t_i
- evolution is **time symmetric**: you can apply $\mathfrak{U}^\dagger[t_i]$ to undo the action of a given timestep
- quantum computation will work by
  - placing the computer in an initial state,
  - applying a sequence of unitary operators to the state
  - measuring the output and producing a final state
- the sequence of unitary matrices, i.e. the system dynamics, are determined via the **Schrodinger equation**
  - classical physics gave conservation of energy
  - **Hamiltonian:** $\mathcal{H}$ the observable for energy, with a hermitian matrix representing it
  - solution with initial conditions allows determination of system evolution

$$\frac{\partial\,|\psi(t)\rangle}{\partial t} = -i\frac{2\pi}{\hbar}\mathcal{H}\,|\psi(t)\rangle$$

Summary

- quantum dynamics is given by unitary transformations
- unitary transformations are invertible: all closed system dynamics are time reversible, provided that no measurements are involved
- concrete dynamics is given by the Schrodinger equation, which determines the evolution of a quantum system whenever its hamiltonian is specified

Assembling Quantum Systems

Assembly

- consider a system with 2 particles confined to the grid,

  - positions of particle 1 can be $\{x_0, ..., x_{n-1}\}$
  - positions of particle 2 can be $\{y_0, ..., y_{m-1}\}$

- assembling quantum systems means tensoring the state space of their constituents
- **Postulate:** assume we have 2 independent quantum systems Q, Q' represented by respective vector spaces $\mathbb{V}, \mathbb{V}'$.
  The quantum system obtained by merging Q and Q' will have the tensor product $\mathbb{V} \otimes \mathbb{V}'$ as a state space
- using this postulate, we can assemble as many systems as we want: the tensor product is associative, so we can build progressively larger systems

$$\mathbb{V}_0 \otimes ... \otimes \mathbb{V}_k$$

- by considering the electromagnetic field as a system composed of infinitely many particles, you can use this procedure to make field theory amenable to the quantum approach
- considering the confined particle example, there are nm basic states: $|x_i\rangle \otimes |y_j\rangle$ means particle 1 is at x_i, particle 2 is at y_j
- you can then express the generic state vector as a superposition of the basic states:

$$|\psi\rangle = \sum_{i,j} c_{ij} |x_i\rangle \otimes |y_j\rangle$$

- this is a vector in the nm dimensional complex space $\mathbb{C}^{mn}$
- $|c_{ij}|^2$ gives the probability of finding the two particles at x_i, y_j respectively

Entanglement

- the basic states of the assembled system are the tensor product of basic states of its constituents
- it would be nice if we could rewrite an arbitrary state vector as the tensor product of two states from respective subsystems
- this cannot be done (in general). Consider the following 2 particle system in state $|\psi\rangle$

$$|\psi\rangle = |x_0\rangle \otimes |y_0\rangle + |x_1\rangle \otimes |y_1\rangle = 1 |x_0\rangle \otimes |y_0\rangle 0 |x_0\rangle \otimes |y_1\rangle + 0 |x_1\rangle \otimes |y_0\rangle + |x_1\rangle \otimes |y_1\rangle$$

- attempt to write $|\psi\rangle$ as the tensor product of 2 states from respective subsystems

$$(c_0 |y_0\rangle + c_1 |y_1\rangle) \otimes (d_0 |y_0\rangle + d_1 |y_1\rangle) = c_0 d_0 |x_0\rangle \otimes |y_0\rangle \; c_0 d_1 |x_0\rangle \otimes |y_1\rangle + c_1 d_0 |x_1\rangle \otimes |y_0\rangle + c_1 d_1 |x_1\rangle \otimes |y_1\rangle$$

- This would imply $c_0 d_0 = c_1 d_1 = 1$ and $c_0 d_1 = c_1 d_0 = 0$, which has no solutions. Therefore $|\psi\rangle$ cannot be written as a tensor product
- what does this mean? If you measure the first particle, you have a 50% chance of finding it at position x_0. If it is found at x_0, then, as $|x_0\rangle \otimes |y_1\rangle$ has coefficient 0, there is no chance of finding particle 2 at position y_1: i.e. particle 2 must be at position y_0!
- we say that the individual states of the particles are **entangled**
- this holds even if x_i is light years away from y_i: regardless of spatial distance, a measurement's outcome for one particle will always determine the measurement's outcome for the other one
- other states are perfectly able to be decomposed into tensor products of subsystem states, and these are referred to as **separable** states

Spin

- there is a law of conservation of total spin of quantum system
- consider z direction, and the corresponding spin basis, up and down
- consider a composite particle whose total spin is 0: the particle may split up at some point into 2 particles that have non-zero spin
- the spin states of those two particles will then be entangled: the sum of the spins must cancel each other out to conserve total spin
- if we measure the z-direction spin of the left particle in state up ($|\uparrow_L\rangle$), the spin of the right particle must be $|\downarrow_R\rangle$
- the bases for the left and right particles are
  - $\mathcal{B}_L = \{\uparrow_L, \downarrow_L\}$
  - $\mathcal{B}_R = \{\uparrow_R, \downarrow_R\}$
- the basis for the entire system is:

$$\{\uparrow_L \otimes \uparrow_R, \uparrow_L \otimes \downarrow_R, \downarrow_L \otimes \uparrow_R, \downarrow_L \otimes \downarrow_R\}$$

- the entangled particles can then be described by:

$$\frac{|\uparrow_L \otimes \downarrow_R\rangle + |\downarrow_L \otimes \uparrow_R\rangle}{\sqrt{2}}$$

- when you measure the left particle and it collapses to the state $|\uparrow_L\rangle$, instantaneously, the right particle collapses to the state $|\downarrow_R\rangle$ even if it is millions of light years away

Summary

- the tensor product allows us to build complex quantum systems out of simpler ones
- the new system is not able to be analysed simply in terms of states of the subsystems: an entire set of new states has been created which cannot in general be resolved into their constituents
- entanglement is used in quantum computing for:
  - algorithm design
  - cryptography
  - teleportation
  - decoherence