Modelling System Dynamics with Graphs

- a weighted digraph can be represented as an adjacency matrix
- classical deterministic systems can be modelled without weights (i.e. all weights are 0 or 1)
- classical probabilistic systems can be modelled with real weights
- quantum systems can be modelled with complex weights

Classical Deterministic Systems

- e.g. marbles moving between vertices
- the states of a system correspond to column vectors (state vectors) $\mathbf{x}$
- dynamics of a system as a digraph with weight whose weights are in $0, 1$ have corresponding matrix M
- the progression from one state to another in one time step, multiply the state vector by a matrix $M\mathbf{x}$
- multiple step dynamics are obtained via matrix multiplication $M^k\mathbf{x}$

Classical Probabilistic Systems

- in quantum mechanics
  - there is inherent **indeterminacy** in our knowledge of a physical state
  - states change according to **probabilistic** laws: the laws governing a system's evolution are given by describing how states transition from one to another with a certain likelihood

Adjacency Matrix

- e.g. marbles moving between vertices with some probability
- to capture probabilistic scenarios, the state of the system corresponds to the probability e.g. of a marble being on a vertex
- weights therefore are real-valued numbers between 0 and 1
- corresponding adjacency matrix is **doubly stochastic:** sum of each row and sum of each column is 1

Time Symmetry

- row vector $\mathbf{w}$ also corresponds to a state of the system

  - $\mathbf{w}M = \mathbf{z}$

- the transpose of M, M^T corresponds to the original digraph with reversed arrows
- this is akin to travelling back in time
- left multiplication of M takes states from time t to $t + 1$
- right multiplication of M takes states from time t to $t - 1$
- **time symmetry** of quantum mechanics is important
- system dynamics are entirely symmetric: replacing column vectors with row vectors, and forward evolution with backward evolution, the laws of dynamics still hold

Summary

- the vectors representing states of a probabilistic system express indeterminacy about the exact physical state
- matrices representing dynamics express indeterminacy about how the system will change over time
- the matrix entries allow calculation of likelihood of transitioning from one state to the next
- the progression of the system is simulated by matrix multiplication

Quantum Systems

Interference

- in quantum systems, a weight is represented by a normalised complex number c, such that $|c|^2$ is a real number between 0 and 1.
- what is the difference between using real probabilities directly and indirect probabilities (via complex numbers)? **interference**
  - real number probabilities can **only** increase when added
  - e.g. $p_1, p_2 \in [0, 1] : (p_1 + p_2) \geq p_1 \wedge (p_1 + p_2) \geq p_2$
  - complex numbers can cancel each other out and lower their probability
  - e.g. $c_1, c_2 \in \mathbb{C}$. $|c_1 + c_2|^2$ is not necessarily bigger than $|c_1|^2, |c_2|^2$

Adjacency Matrix

- in quantum realm, graphs are represented by matrices with complex entries
- rather than doubly stochastic, adjacency matrices are **unitary**, i.e. $U^\dagger U = I = UU^\dagger$
- the element-wise squared modulus of a unitary matrix is doubly stochastic

  - i.e. if U is unitary with elements u_{ij}, then the matrix with elements $|u_{ij}|^2$ is doubly stochastic

- from the graph-theory perspective, if U is the unitary matrix taking a state from t to $t+1$, then $U^\dagger$ is the matrix taking a state from t to $t-1$
- consider the following sequence of operations:

$$\mathbf{v} \to U\mathbf{v} \to U^\dagger U\mathbf{v} \to I\mathbf{v} = v$$

- you get the identity matrix: in graph terms this means "stay where you are". $U^\dagger$ undoes the action of U, leaving you with probability 1 where you started

Double Slit

- probability of measuring photon at centrepoint classically: non-zero
- interference on the wall at the centrepoint of slits: 0 probability of photon at this location, even if the experiment was conducted with a single photon
- this suggests interpretation of the state vector as representing the probabilities of the photon being at a particular state is inadequate
- to have some state vector suggests that the photon is in all states simultaneously: the photon passes through both slits simultaneously, and when it does so it can cancel itself out
- photon is in a **superposition** of states
- the reason we see particles in one position is because we have performed a **measurement**
- new definition of state: a system is in state $\mathbf{x}$ if after measuring it, it will be found in position i with probability $|c_i|^2$
- superposition of states is the power behind quantum computing: while classical computers are in a single state at any moment, consider putting a computer in all states at once - lots of parallel processing

  - only possible in the quantum realm

Summary

- states in a quantum system are represented by column vectors of complex numbers whose sum of moduli squared is 1
- the dynamics of quantum systems is represented by unitary matrices, and is therefore reversible
- undoing is obtained via algebraic inverse: the adjoint of the unitary matrix which represents forward evolution

- probabilities of quantum mechanics are given as the modulus square of complex numbers
- quantum states can be superposed: a physical system can be in more than one basic state simultaneously

Assembling Systems

- consider composite classical probabilistic systems, with results applicable to quantum systems
- composite systems: e.g. red marble follows graph G_R, and blue marble follows graph G_B, with corresponding adjacency matrices A, B
  - state for the two-marble system is the **tensor product** of the state vectors of each system
  - dynamics for the two-marble system is the **tensor product** of the adjacency matrices: this corresponds to the Cartesian product of 2 weighted digraphs

Entangled States

- in the quantum world there are many more possible states than just states that can be combined from smaller ones
- **entangled states** are those that are not the tensor product of smaller states
- there are also many more possible actions on a combined quantum system than simply that of the tensor product of individual system's actions

Exponential growth

- Cartesian product of n vertex graph with p vertex graph is an np vertex graph
- if you have an n vertex graph G with m different marbles, you need to look at the graph

$$G^m = G \times G \times ...G$$

which has n^m vertices - if M_G is the associated adjacency matrix, we will be interested in

$$M_G^{\otimes m} = M_G \otimes M_G \otimes ... \otimes M_G$$

which is an n^m-by-n^m matrix

- consider a bit as a 2-vertex graph with a marble on the 0 vertex/1 vertex
- to represent m bits, each with a single marble, one would need a 2^m vertex graph, with a 2^m-by-2^m matrix

- this means exponential growth in resources needed for the number of bits under discussion
- this was the motivator for Feynman to start discussing potential of quantum computing