Complex Systems

What is a system?

- set of things working together as parts of a mechanism/interconnecting network
- complex whole
- e.g. physiology: set of organs in body with common structure/function
- e.g. biology: human body as a whole
- e.g. automobile

What makes a system complex?

- consider:
  - how many parts in the system?
  - are rules of parts simple or complicated?
  - is the behaviour of the system as a whole simple or complex?
- few parts, simple rules, simple behaviour: e.g. 2 body problem
  - solved analytically
  - produces regular trajectories, predictable
  - complete understanding
- few parts, simple rules, complex behaviour:
  - e.g. 3 body problem
    * solved numerically
    * chaotic trajectories
  - e.g. logistic equation: simple dynamical equation giving rise to broad range of complex emergent behaviours, including chaos
- many parts, simple rules, simple behaviour
  - e.g. crystals: highly ordered and regular
  - e.g. gases: highly disordered but statistically homogeneous
- many parts, simple rules, complex behaviour
  - e.g. flocking behaviour
  - e.g. cellular automata
  - e.g. complex networks

- many parts, complicated rules, complex behaviour
  - e.g. biological development, evolution, societies, markets
    * heterogeneous rules, specialisation, hierarchies
    * complex behaviour but reproducible and robust
    * e.g. termite colony producing a mound: local interactions with one another mediated by environment, no centralised leader
    * e.g. division of a fertilised cell -> increased specialisation -> organs -> full organism
    * e.g. brain, immune system
- many parts, complicated rules, deterministic behaviour
  - classical engineering: many specialised parts, globally designed to ensure predictable behaviour
  - behaviour is designed not emergent
- many parts, complicated rules, centralised behaviour
  - e.g. orchestra: global behaviour is emergent but not complex due to centralised controller

What is a complex system?

- complex systems
  - made of a number of components
  - that interact
  - typically in a non-linear fashion
  - may arise from/evolve through self-organisation
  - neither completely regular nor completely random
  - permit development of emergent behaviour at macroscopic scale

Properties

Emergence

- system has properties individual parts do not
- properties are not easily inferred/predicted
- different properties can emerge from the same parts depending on context/arrangement
- e.g. bird flock: global pattern of movement (flock shape) emerges, not apparent from observation of any individual bird

Self-organisation

- order increases without external intervention as a result of interactions between parts
- e.g. bird flock: global pattern: neither completely regular nor completely random but does show some order
    - order is not imposed externally, but results from simple set of rules applied by individuals based on local context

Decentralisation

- no single controller
- **distribution:** each part carries a subset of global information
- **bounded knowledge:** no part has full view of the whole
- **parallelism:** parts can act simultaneously
- e.g. bird flock
    - bird at front is not leader
    - all birds act independently based on local cues
    - no bird is aware of whole flock
    - all birds are continuously updating their position

Feedback

- positive feedback: amplify fluctuations in system state
    - e.g. bubbles in cryptocurrency
    - decreases system stability
- negative feedback: damp fluctuations in system state
    - e.g. air-con thermostat
    - increases system stability

What is a model?

- simplified description of a system/process
    - typically mathematical
    - assists calculations/predictions

Why build models?

- examine system behaviour in a way infeasible in real world
  - too expensive
  - too time consuming
  - unethical
  - impossible
- allow us to understand a system by building it
  - analyse
  - predict
  - understand

Mathematical models

Macro-Equations

- **macro equations:** describe global state/behaviour of system, ignoring individual components
  - e.g. predator-prey system described with ODEs
- **analytical solution:** closed form solution found via calculus of macro-equations
- **numerical solution:** discretisation of time/space
  - as typically the case, no closed form solution exists
  - solve algorithmically to discover future trajectory

Computational Models

No macro-equations

- cannot formulate global description of system: pressure, GDP, flock
- systems contain heterogeneity:
  - **differentiated** parts
  - irregularly **located** parts
  - parts connected in complex **network**
- systems are dynamically adaptive
  - interaction topology changes over time in response to environment

Agent-based models (ABMs)

- arose in 1960s to model systems too complex for analytical descriptions
- system parts: agents with local state and rules
- system structure: pattern of local interaction between agents
- system behaviour: dynamic rules for updating agent state on basis of interactions

Steps in modelling complex system

1. define key questions
2. identify structure - parts and interactions - of system
3. define possible states for each part
4. define how state of each part changes over time through interactions
5. verify, validate, evaluate model: simplicity, correctness, robustness
6. define, run experiments to address key questions

Questions in complex system modelling

- how to explain current/past events?
  - disease outbreak
  - climate change
  - mass extinction
  - market crashes and bubbles
  - collapse of civilisation
- how to predict future behaviour?
  - motivates understanding of system
- how to design/build better engineered systems
  - nature inspired optimisation - ant colonies, particle swarms
  - decentralised computing, Internet
  - autonomous sensor networks

Dynamical systems and Chaos

What is a dynamical system?

- **state space $\{x_i\}$:** set of possible states

- time t: discrete/continuous
- **update rule:** state at present time x_t as function of earlier states
  - deterministic: history uniquely determines present state
  - non-deterministic: probabilistic/stochastic
- **initial condition** x_0**:** state at $t = 0$

Functions and iteration

- iteration: using output of previous function application as input for next function application

$$x_0, x_1 = f(x_0), x_2 = f(f(x_0)), \dots$$

Population growth

- apex predator: red foxes, not predated by other species
- prey: e.g. numbat, bilby

Fox population

- assumptions:
  - initial population: 2 female red foxes
  - female red fox reproduces in 1st year of life
  - female red fox reproduces once in its life
  - half of newborn kits are female
- number of female red foxes doubles each year: $x_{t+1} = 2x_t$
  - x_t number of female red foxes alive in year t
- produces exponential growth: $x_t = x_0 2^t$
- more generally $x_t = x_0 r^t$, where r governs steepness of curve
  - r: intrinsic rate of increase
  - e.g. $r = 1$: replacement fertility; stable population
  - e.g. $r < 1$: shrinking population
- **orbit/trajectory:** sequence of states visited as dynamical system evolves over time

Model refinement: logistic model of population growth

- unlimited exponential growth unrealistic: red foxes will at some point run out of food/space
- extra assumptions:
  - few red foxes: plenty of food, rapid growth
  - many red foxes: not enough food, slower growth
- define population size A at which foxes eat all available food, producing starvation and 0 foxes next year
  - A: carrying capacity

$$P_{t+1} = rP_t(1 - \frac{P_t}{A})$$

- P ~ A: fewer foxes next year; $(1 - P_t/A) \approx 0$
- P ~ 0: more foxes next year; $(1 - P_t/A) \approx 1$
- let $x = \frac{P}{A}$: **logistic map**

$$x_{t+1} = rx_t(1 - x_t)$$

Logistic Map

$$x_{t+1} = rx_t(1 - x_t)$$

- rx_t: positive feedback - $(1 - x_t)$: negative feedback - display range of behaviours depending on the value of r - **fixed point:** - numerically: $x_{t+1} = x_t$

$r \in [0, 1]$

- population dies out
- 0: fixed point, stable for $r \in [0, 1]$

$r = 0.9$

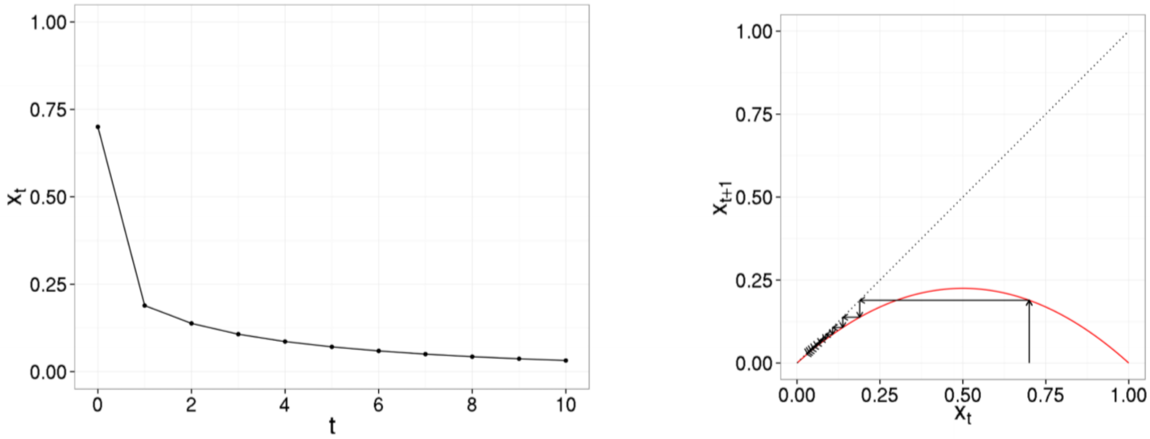

Figure 1: r = 0.9

$r \in [1, 3)$

- no longer attracted towards 0
- new stable fixed point: identify as intersection of parabola with identity line
- system moves towards stable fixed point whether x_0 less than/greater than it
- i.e. population has a stable value it likes to remain at
- fixed point at 0 still exists but is unstable: at locations near 0, system moves away from 0
- only if x_0 is exactly 0, the system will stay at 0 indefinitely

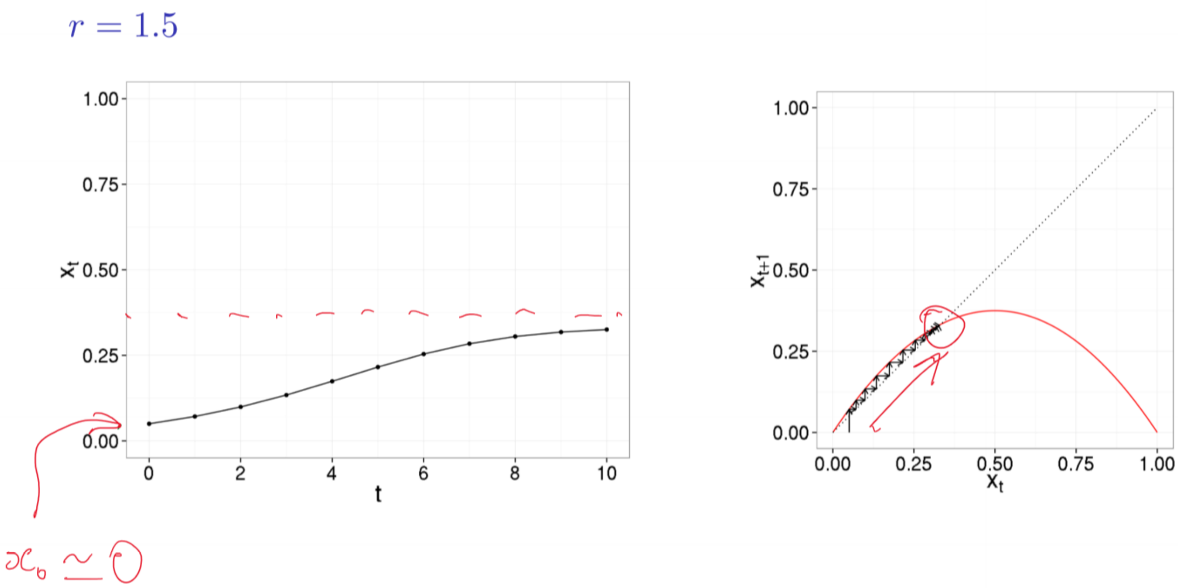

Figure 2: r = 1.5

$r \geq 3$

- no stable fixed points
- @ $r = 3.2$: limit cycle with period 2; oscillation between 2 values
- @ $r = 3.52$: limit cycle with period 4
- @ $r = 3.56$: limit cycle with period 8
- **bifurcation:** qualitative change in system's behaviour
  - e.g. transition from fixed point to limit-2 cycle
  - indicate points in parameter space where system behaviour changes dramatically
  - tipping point
- @ $r = 3.84$: initial transient, followed by limit cycle with period 3

r = 3.84

$r = 4.0$**: Chaotic attractor**

- system behaviour appears all transient
- system displays aperiodic behaviour typical of deterministic chaos
- can be shown that time series never repeats

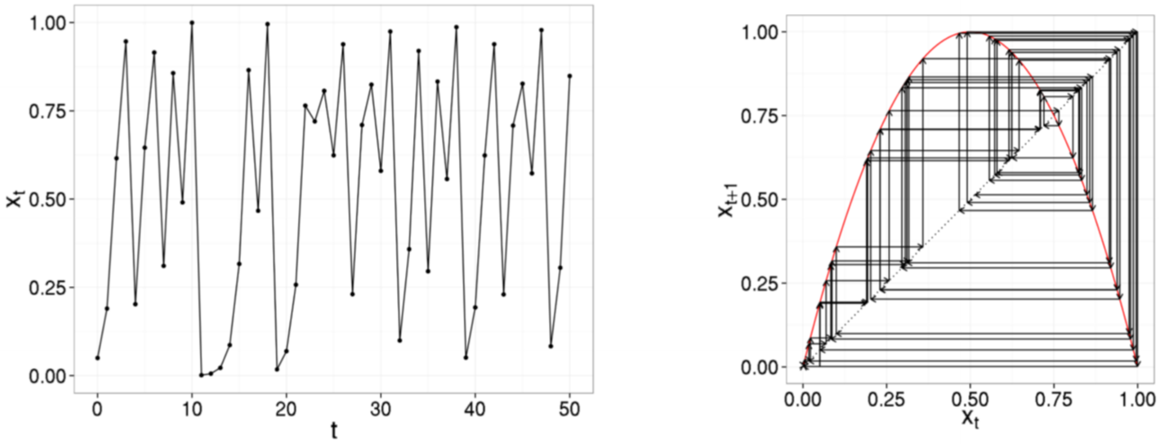

Figure 3: r = 4

Chaos

A system is **chaotic** if it displays **all** of the following properties

1. **deterministic** update rule: not random; given same starting conditions, get same result
2. **aperiodic** system behaviour: trajectory does not repeat
3. **bounded** system behaviour: exponential never repeats but aperiodicity is not interesting; logistic map bounded between 0 and 1
4. **sensitivity** to initial conditions: butterfly effect; wildly varying outputs with only small modification to inputs

Bifurcation Diagrams

- for logistic map, sweep through parameter space of r
- output: series of values system converges to after initial transient

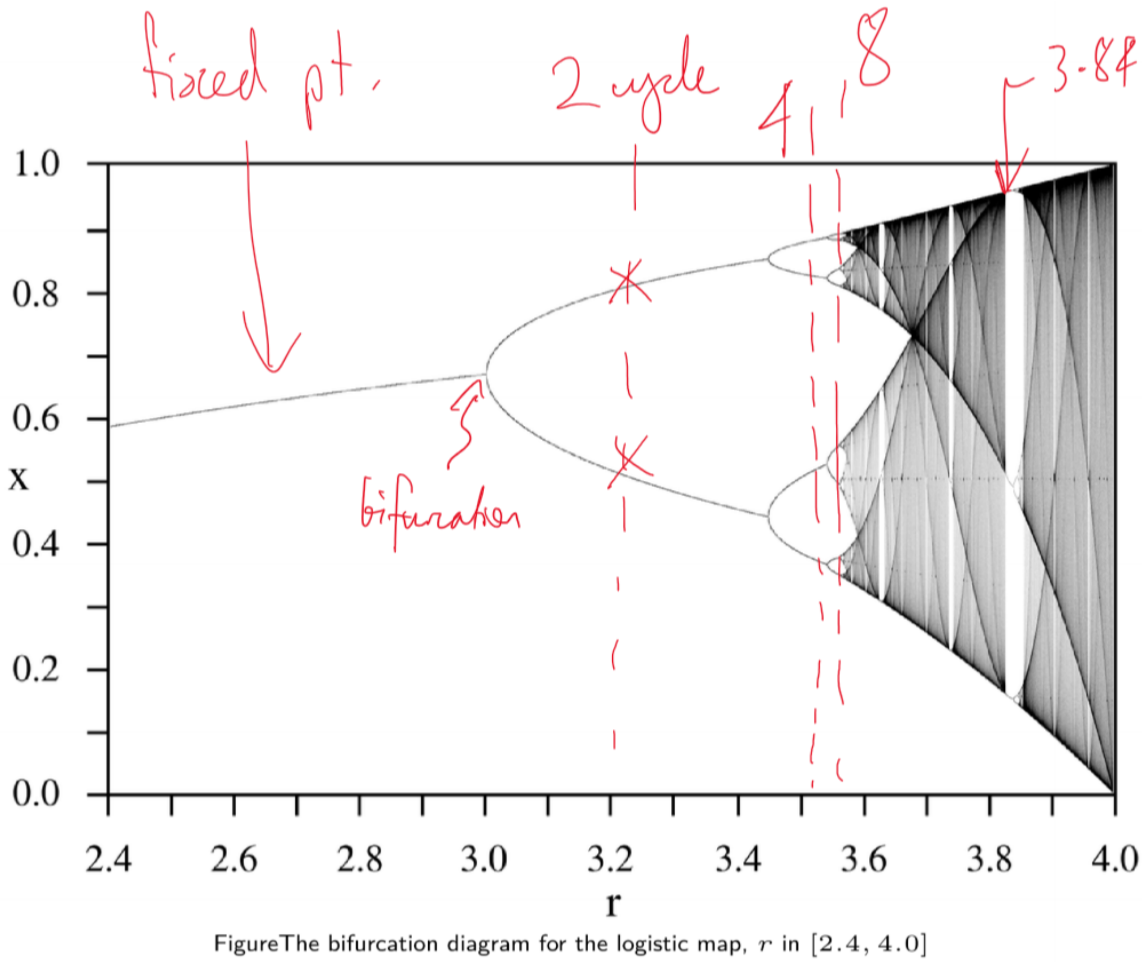

Figure 4: bifurcation diagram of logistic map

- bifurcation diagram of logistic map has **fractal/self-similar** properties: i.e. if you zoom in you see the structure repeated