Divide and Conquer

Table of Contents

Overview

- divide and conquer:
  - split into multiple smaller problems
  - solve these: typically recursive, and may become brute force when sufficiently small
  - combine sub-problem results to get final solution

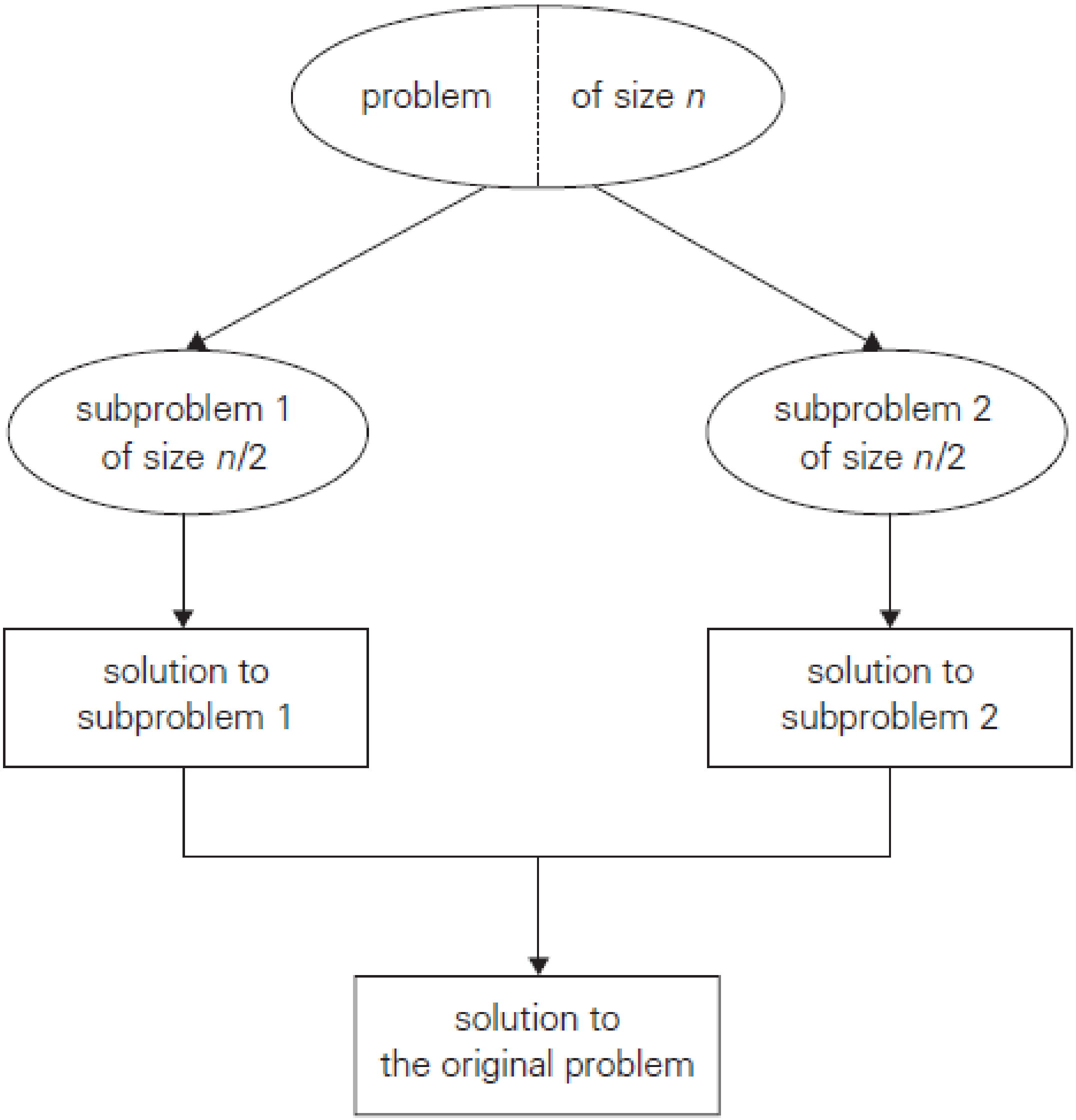


Figure 1: divide_and_conquer

- not necessarily more efficient than brute force
- some divide and conquer algorithms are the most efficient algorithms possible
- well suited to parallel computation, where each subproblem is solved simultaneously on a distinct processor

General divide and conquer recurrence relation

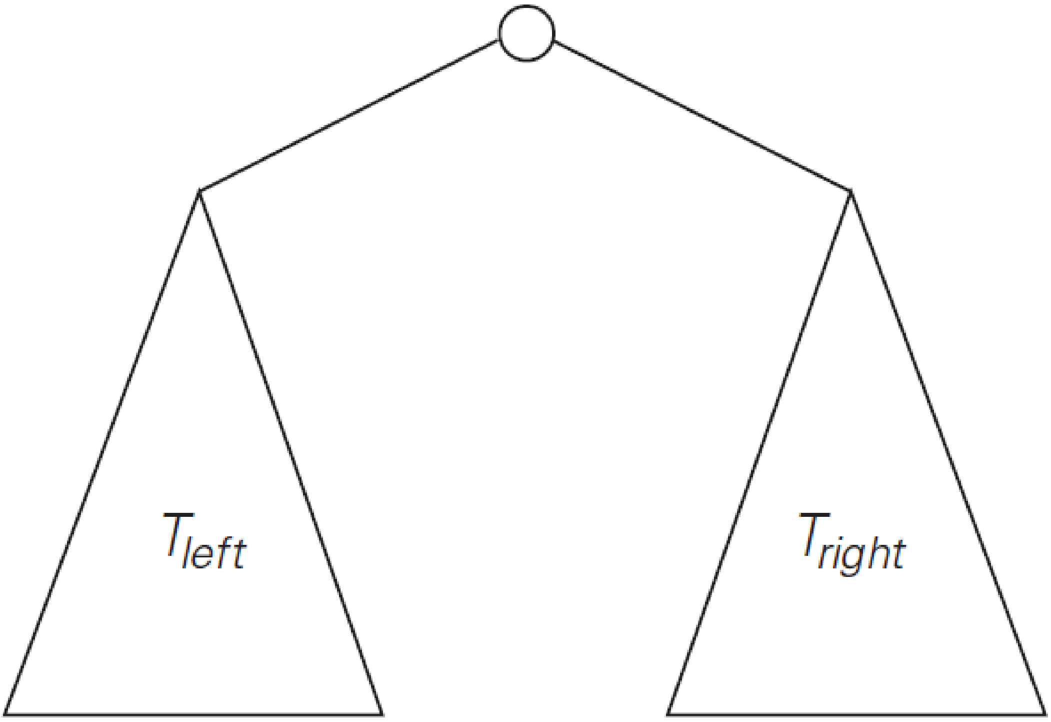
- problem size n can be divided into b sub-problems of size n/b, with a sub-problems needing to be solved
  - i.e. $a \geq 1, b > 1$, with a, b constants
- with $n = b^k$ for some $k \in \mathbb{Z}^+$: time complexity $T(n)$

$$T(n) = aT(\frac{a}{b}) + f(n)$$

- $f(n)$: time spent dividing into subproblems and combining subproblem solutions
- applying the master theorem: if $f(n) \in \Theta(n), k > 0$:

$$T(n) \in \begin{cases} \Theta(n^k) \text{ if } a < b^k \\ \Theta(n^k \log n) \text{ if } a = b^k \\ \Theta(n^{\log a_b)} \text{ if } a > b^k \end{cases}$$

Binary Tree



- Binary tree T: finite set of nodes; a root + 2 disjoint binary trees T_L (left) and T_R; otherwise empty

- all subtrees are also binary trees: many problems can be approached with divide and conquer/recursive algorithms
- not all questions about binary trees require traversal of entire tree
  - e.g. search and insert requires processing one of two subtrees

Height

- height: length of longest path from root to leaf

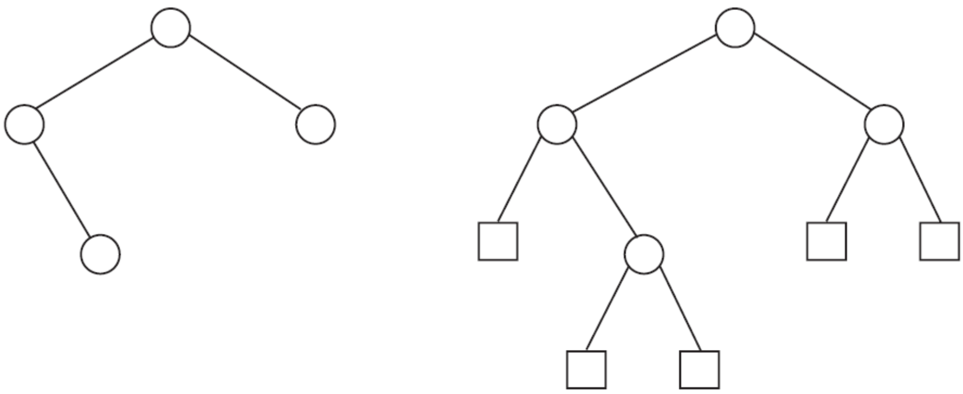
```
"""
Recusively compute height of binary tree
input: binary tree T
output: height of T
"""
Height(T):
    if T is empty:
        return -1
```

```
    else:
        return max(Height(T_L), Height(T_R) + 1
```

- measure instance size by number of nodes $n(T)$
- number of comparisons for max height will be the same as number of additions $A(n(T))$, so for $n(T) > 0$ with $A(0) = 0$:

$$A(n(T)) = A(n(T_L)) + A(n(T_R)) + 1$$

- *checking* the tree is empty is actually the most common operation here
- consider the tree drawn with **internal nodes** (circles) and empty children as **external nodes**



(a) (b)

(rectangles)
- $\implies$ comparison to empty set occurs for *all* internal and external nodes, while addition is only for internal nodes
- for a full binary tree with n internal nodes, every node except the root is 1 of 2 children
- total internal + external nodes is then:

$$n + x = 2n + 1$$

So

$$x = n + 1$$

Number of comparisons to empty tree $C(n)$ is then:

$$C(n) = n + x = n + 1$$

Number of additions $A(n)$ is:

$$A(n) = n$$

Tree traversals

- most import divide and conquer algorithms for trees are tree traversals
- **preorder traversal**: root $\rightarrow$ left subtree $\rightarrow$ right subtree
- **inorder traversal**: left subtree $\rightarrow$ root $\rightarrow$ right subtree
- **postorder traversal**: left subtree $\rightarrow$ right subtree $\rightarrow$ root

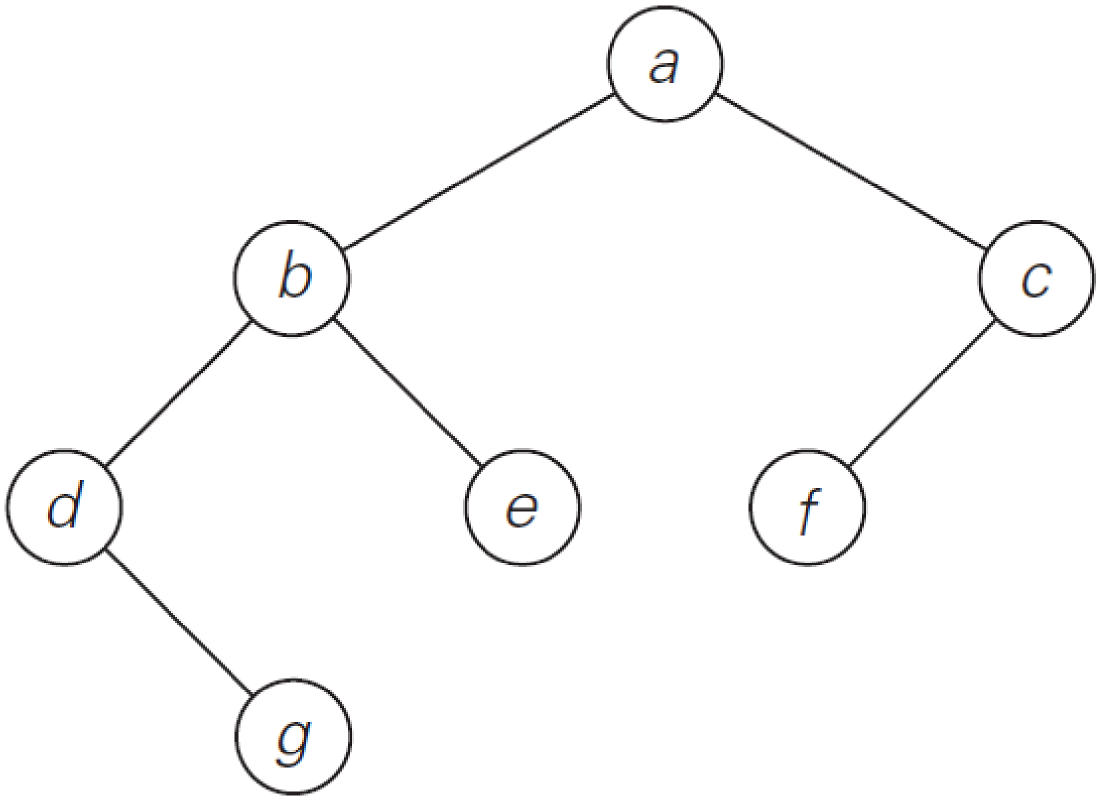


Figure 2: tree_traversals

- preorder: a, b, d, g, e, c, f
- inorder: d, g, b, e, a, f, c
- postorder: g, d, e, b, f, c, a

- efficiency: identical to that of height, as recursive calls are made for each node of extended binary tree

Closest Pair

- brute force closest pair: $\Theta(n^2)$
- P: set of $n > 1$ distinct points in Cartesian plane, ordered by x-coordinate (non-decreasing)
- Q: set of $n > 1$ distinct points in Cartesian plane, ordered by y-coordinate (non-decreasing)
- if $2 \leq n \leq 3$: solve by brute force
- $n > 3$:
  - divide points into 2 subsets P_l, P_r of size $\lfloor n/2 \rfloor$, analogous to vertical line through median x-coordinate
  - solve recursively for P_l, P_r, to produce minimum distance d_l, d_r
  - combine result: $d = \min d_l, d_r$
  - now need to consider whether there are any pairs of points $p_l \in P_l, p_r \in P_r$ such that $dist(p_l, p_r) < d$; i.e. are there any points between P_l, P_r closer than d that we missed by splitting up the problem?
  - only need to consider points from a strip of width $2d$ around the median, call these points S obtained from Q
  - scan S, looking for any points closer than d_min:
    * for a point $p'(x', y') to be closer to $p(x, y)$ than d_min, the point must follow p
    * i.e. $|y - y'| < d_min$
    * so p' must be in a rectangle with width $2d$ and height d_min
    * rectangle can only contain a few points as points in left and right rectangle must be at least d distance apart
    * can be shown that rectangle has ≤ 8 points (more rigorously ≤ 6)
    * algorithm considers no more than 5 next points on list S before moving to the next p
- linear time for dividing problem in two
- linear time for combining solutions
- if n is a power of 2:

$$T(n) = 2T(\frac{n}{2}) + f(n)$$

where $f(n) \in \Theta(n)$ Applying master theorem, with $a = 2, b = 2, d = 1$:

$$T(n) \in \Theta(n \log n)$$

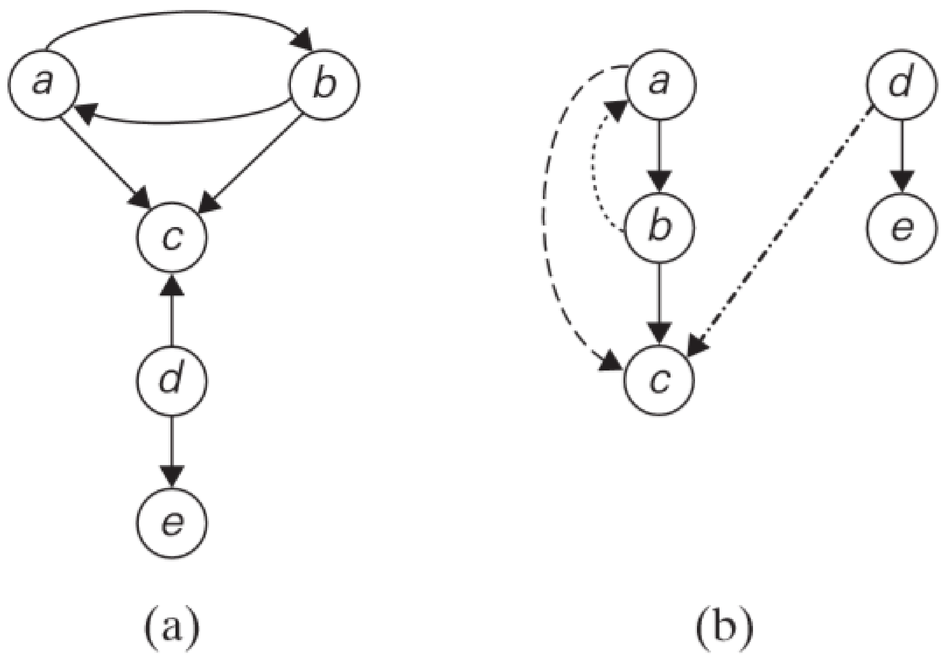
- best efficiency possible for this problem: can be shown that any algorithm for this problem is $\Omega(n \log n)$

```
EfficientClosestPair(P, Q):
# solve closest pair using divide and conquer
# input: array P of n >= 2 points sorted in nondecreasing order in x-coord
#        array Q in >= 2 points sorted in nondecreasing order in y-coord
# output: euclidean distance between closest pair of points
if n <= 3:
    return BruteForceClosestPair(P, Q)
else:
    copy first ceil(n/2) points of P to array P_l
    copy same ceil(n/2) points of Q to array Q_l
    copy remaining floor(n/2) points of P to array P_r
    copy same floor(n/2) points of Q to Q_r
    d_l = EfficientClosestPair(P_l, Q_l)
    d_r = EfficientClosestPair(P_r, Q_r)
    d = min(d_l, d_r)
    m = P[ceil(n/2)-1].x
    copy all points of Q for which abs(x-m) < d into S[0..num-1]
    dminsq = d^2
    for i = 0 to num-2:
        k = i + 1
        while k <= num-1 and (S[k].y-S[i].y)^2 < dminsq:
            dminsq = min((S[k].x-S[i].x)^2+(S[k].y-S[i].y)^2, dminsq)
            k += 1

    return sqrt(dminsq)
```

Topological Sorting

- digraph traversal can be performed with DFS and BFS, but the structure of the forests this yields can be much more complex than for an undirected graph
- DFS forest for a digraph can have (referring to digraph below)
  - tree edges: (ab, bc, de)
  - back edges: (ba)
  - forward edges: (ac)
  - cross edges: (dc)



(a) Digraph (b) DFS forest of digraph for DFS traversal started at a

- **directed cycle**: sequence of 3+ vertices which are connected as ordered, starting and ending on the same vertex
- presence of back-edge on DFS forest $\Rightarrow$ digraph has directed cycle
- **dag/directed acyclic graph**: digraph with no directed cycles
- **topological sort**: find an order of vertices such that for every edge in the graph, the start vertex is listed before the end vertex
  - solution exists $\iff$ graph is a dag

DFS topological sort

- perform DFS traversal
- note the order in which vertices become dead ends, such that they are popped off the traversal stack
- the reverse order of this is a solution to the topological sort
- if a back edge is encountered, the graph is not a dag, so a topological sort is impossible
- to understand why this works: when a vertex v is popped off the DFS stack, no vertex u with an edge (u, v) can be among the vertices popped off before v. If there was such a vertex, (u, v) would be a back edge. This implies u will be listed after v in the popped-off order list, and before

v in the reversed list