Algorithms

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Algorithms

- Sequence of *unambiguous* instructions for solving a problem to obtain required output for *legitimate input* in a *finite* amount of time
- multiple valid solutions with different efficiency

Greatest common divisor

Euclid's algorithm `gcd(m, n)= gcd(n, m mod n)`

For example

```
gcd(24, 60) = gcd(60, 24)
            = gcd(24, 12)
            = gcd(12, 0)
            = 12
```

Since `gcd(m, 0)= m`

Sieve of Eratosthenes

- algorithm to generate consecutive primes not exceeding a given integer `n > 1`
- procedure:
  - generate a list of prime candidates from 2 to n
  - loop over the list, each time eliminating candidates that are multiples of 2, 3, …
  - no pass for 4 is necessary as all multiples of 4 have already been eliminated
  - algorithm continues until no more numbers can be eliminated; remaining numbers are prime
- what is largest `p` whose multiples can still remain on the list to make further iterations of the algorithm necessary?
  - if `p` is a number whose multiples are being eliminated on the current pass, first multiple we should consider is `p.p` because all smaller multiples `2p, ..., (p-1)p` have been eliminated on earlier passes
  - `p.p` should be less than `n` otherwise it isn't a candidate, i.e.

```
p \leq \lfloor\sqrt{n}\rfloor
```

Algorithmic Problem Solving

- understand the problem
- understand the capabilities of the hardware
- decide between exact/approximate solution
- choose design techniques
- design algorithm and data structure

- prove correctness: prove that algorithm yields required result for every legitimate input in finite time
  - often uses mathematical induction
  - for approximation algorithms you need to show error does not exceed defined limit
- analysis
  - time efficiency: run time
  - space efficiency: memory
  - generality
- implement the algorithm

Important problem types

- sorting: rearrange list items in non-decreasing order
  - stable: preserves relative order of equal elements
  - typically algorithms that switch keys far apart are not stable but are faster
  - in-place: doesn't require extra memory to run
- searching: find a given value (*search key*) in a given set
- string processing
  - e.g. string matching
- graph problems
  - graph is a collection of vertices, connected by edges
  - e.g. graph traversal, shortest path
  - graph-coloring: assign smallest number of colors to vertices of a graph such that no two adjacent vertices are the same color (event scheduling)
  - travelling salesman problem: shortest tour through n cities that visits each city only once
- combinatorial problems
  - ask to find a combinatorial object satisfying constraints (e.g. permutation, combination, subset)
  - typically most difficult class of problems: number of objects grows extremely fast with problem size
- geometric problems: points, lines and polygons
  - e.g. computer graphics, robotics, tomography

  - closest-pair problem: given n points in the plane, find the closest pair among them
  - convex-hull problem: smallest convex polygon that contains all points of a set

- numerical problems: mathematical objects of continuous nature

  - solving systems of equations, computing integrals, evaluating functions

Linear data structures

Array

- sequence of n items of the same data type stored contiguously in memory
- accessible by **index**
- each element of an array can be accessed by an identical constant amount of time (c.f. linked lists)
- useful for **strings**

Linked list

- sequence of **nodes** each containing data and **pointers** to other nodes
- **singly linked list**: each node (except last) contains a single pointer to the next element
- nodes are accessed by traversing the list: time dependent on node's location
- doesn't require preliminary reservation of memory
- efficient insertions and deletions
- **header**: special node at start of list, points to first item in list, could contain:

  - metadata about list e.g. current length
  - pointer to last element in list

- **doubly linked list**: each node contains a pointer to the next and previous node

FIGURE 1.3 Array of n elements.

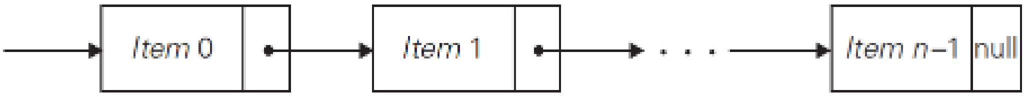


FIGURE 1.4 Singly linked list of n elements.

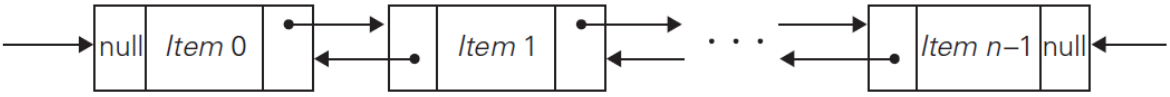


FIGURE 1.5 Doubly linked list of n elements.

List

- **list**: finite sequence of data items
- operations:
  - search for
  - insert
  - delete

Stacks

- **stack**: list in which insertions and deletions are performed at the end (**top**) of the list
  - last-in-first-out
  - picture vertical stack of plates

Queue

- **queue**: elements added to rear, and removed from the front

  - **dequeue**: elements deleted from the **front**
  - **enqueue**: elements added to the **rear**
  - first-in-first-out
  - think queue of customers in line

Priority queues

- *priority queue*: useful for selection of an item of highest priority from dynamically changing candidates
  - collection of data items from a totally ordered universe (e.g. integer/real numbers)
- operations:
  - find largest element
  - delete largest element
  - add a new element
- *heap* is the most efficient solution to this problem

Graphs

- collection of points, called **vertices** or **nodes**, with some connected by **edges**
- a graph

$$G = \langle V, E \rangle$$

  , is a pair of two sets
  - finite nonempty set V, vertices
  - set E of pairs of these items, edges
- if these pairs of vertices is unordered i.e.

$$(u, v)$$

  is the same as

$$(v, u)$$

  , v and u are **adjacent**, connected by **undirected edge**

$$(u, v)$$

- vertices *u* and *v* are **endpoints** of edge

$$(u, v)$$

  - *u* and_v_ are **incident** to this edge (and vice versa)

- a graph is **undirected** if all edges are undirected
- **directed** edge

$$(u, v)$$

  means vertices

$$(u, v)$$

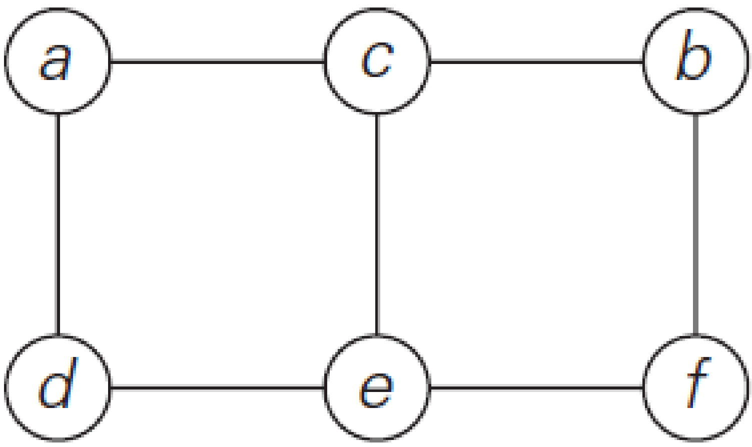
  are not the same as vertices

$$(v, u)$$

  - from **tail** u to **head** v

- a graph is **directed** if all edges are directed (aka **digraphs**)
- convenient to label vertices with letters or numbers
- graph with 6 vertices and 7 undirected edges

```
V = \{a, b, c, d, e, f\}
\newline
E = \{(a,c), (a,d), (b,c), (b,f), (c,e), (d,e), (e,f)\}
```



- digraph with 6 vertices and 8 directed edges

```
  V = \{a, b, c, d, e, f\}
\newline
E = \{(a,c), (b,c), (b,f), (c,e), (d,a), (d,e), (e,c), (e,f)\}
```

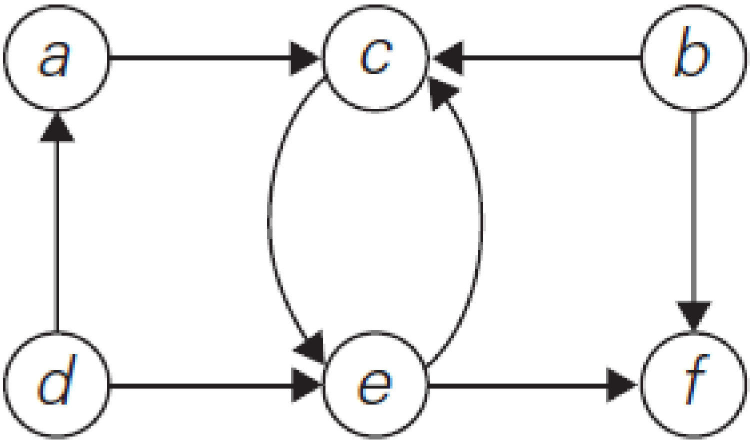


Figure 1: directed_graph

- this definition allows **loops**, including edges connecting vertices to themselves, however unless stated will be expected to have no loops
- definition disallows multiple edges between the same vertices of an undirected graph:
  - number of edges

$$|E|$$

  - number of vertices

$$|V|$$

  -

$$0 \leq |E| \leq |V| \frac{(|V|-1)}{2}$$

- graph is **complete** if every pair of vertices is connected by an edge
  - complete graph with

$$|V|$$

    vertices:

$$K_{|V|}$$

- graph with few missing edges is **dense**

- graph with few edges present is **sparse**

Graph representations

- **adjacency matrix**: for graph with

$$n$$

  vertices is

$$n \times n$$

  boolean matrix
  - row i, col j: 1 if edge from i to j; 0 otherwise
  - undirected graph has a symmetric adjacency matrix

$$A_{ij} = A_{ji}$$

    for all i, j
- **adjacency list**: collection of linked lists for each vertex containing all adjacent vertices (those connected by an edge)
- sparse graphs more efficiently represented by adjacency list
- dense graphs more efficiently represented by adjacency matrix

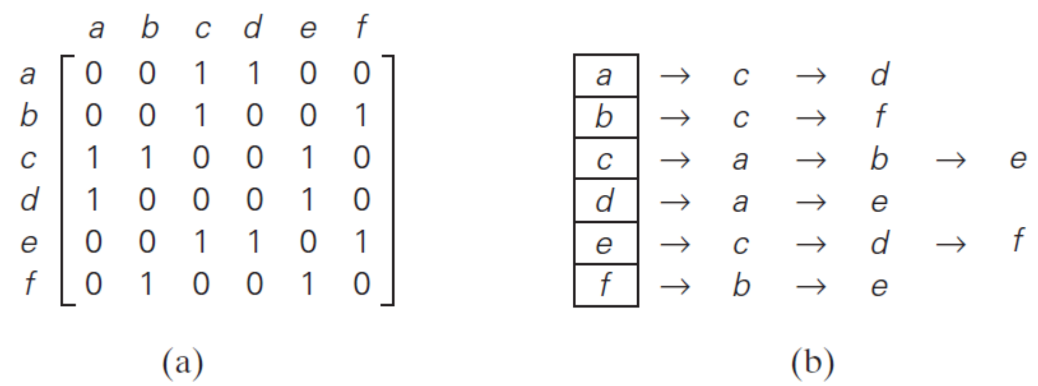


Figure 2: graph_representation

Weighted graphs

- **weighted graph**: graph with numbers (**weights**, **costs**) assigned to edges

- adjacency matrix can be updated to a **weight matrix** such that

$$A_{ij}$$

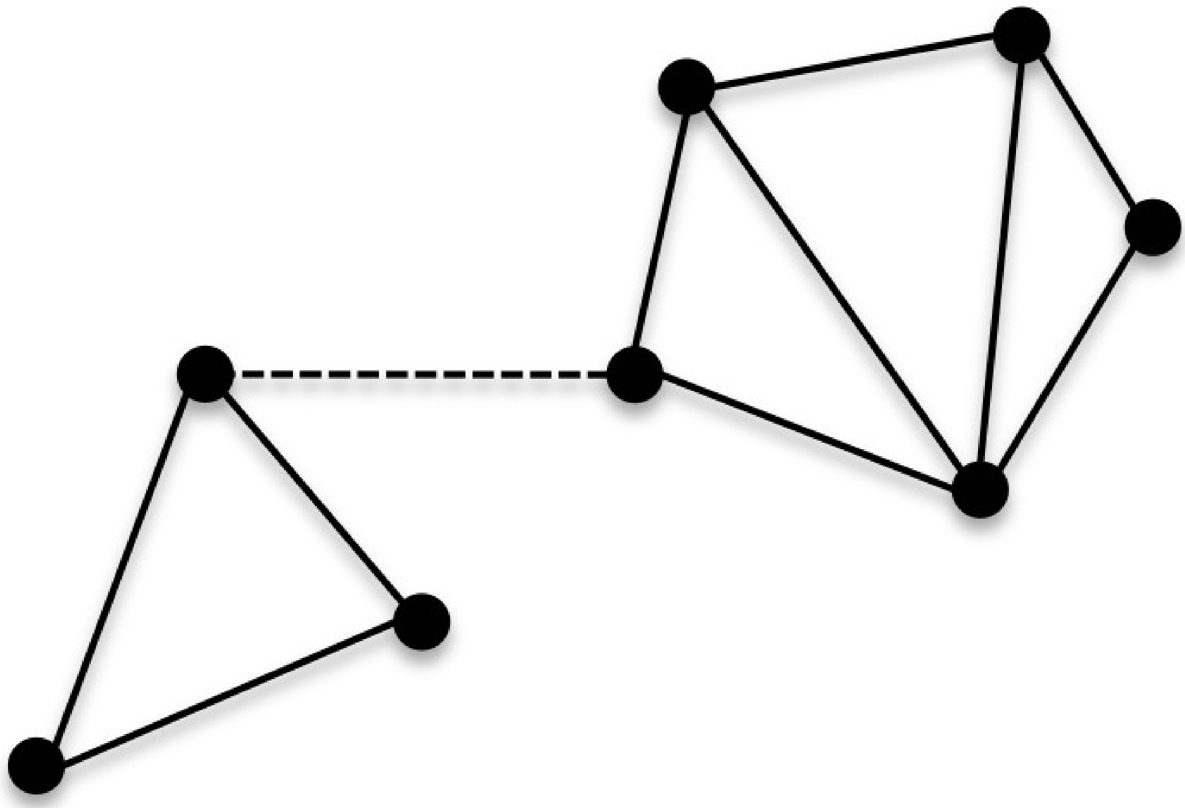
is the weight for that edge

  - if there is no such edge, entries are

$$\infty$$

Paths and Cycles

- **path** from vertex u to vertex v of graph G: sequence of adjacent vertices from u to v.
- **simple path**: all vertices of a path are distinct
- **path length**: (num. vertices)-1, (num. edges)
- **directed path**: sequence of vertices, with each successive pair of vertices u, v having a directed edge (u,v)
- **connected graph**: for every pair of vertices u,v there is a path from u to v
  - i.e. no **unreachable** vertices
- a disconnected graph forms multiple **connected components**: maximal connected subgraphs of a graph



Graph becomes disconnected when dashed line is removed

- **cycle**: path of positive length that starts and ends at the same vertex, without traversing the same edge more than once
- **acyclic**: graph without cycles

Trees

- **free tree**, aka tree: connected acyclic graph
  - Necessary property for graph to be a tree:
    * (number of edges) = (number of vertices) - 1
    * $$| E/ | = | V | -1$$
  - For connected graphs this is a sufficient property; useful for checking if a connected graph has a cycle
- **forest**: graph with no cycles but is not necessarily connected, with each component being called a tree

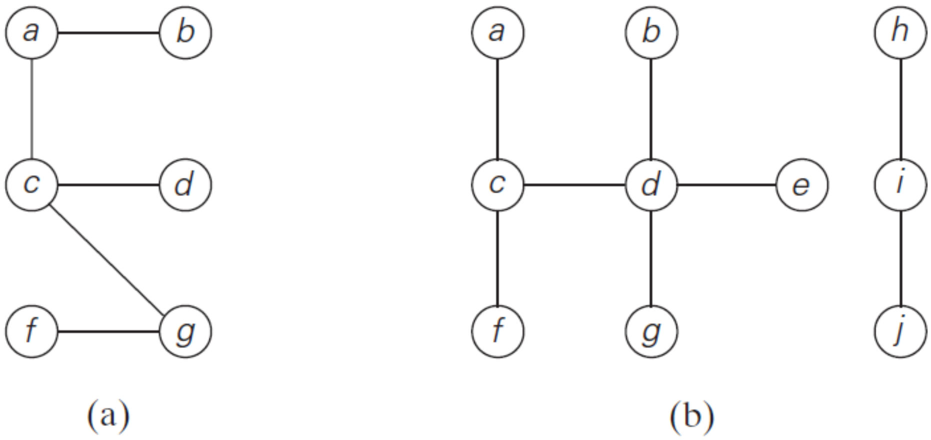


FIGURE 1.10 (a) Tree. (b) Forest.

Figure 3: graph_tree_forest

Rooted trees

- for every two vertices in a tree, there exists exactly one simple path from one vertex to the other
- can select arbitrary vertex in a free tree as **root** of the **rooted tree**
- e.g. file system hierarchy

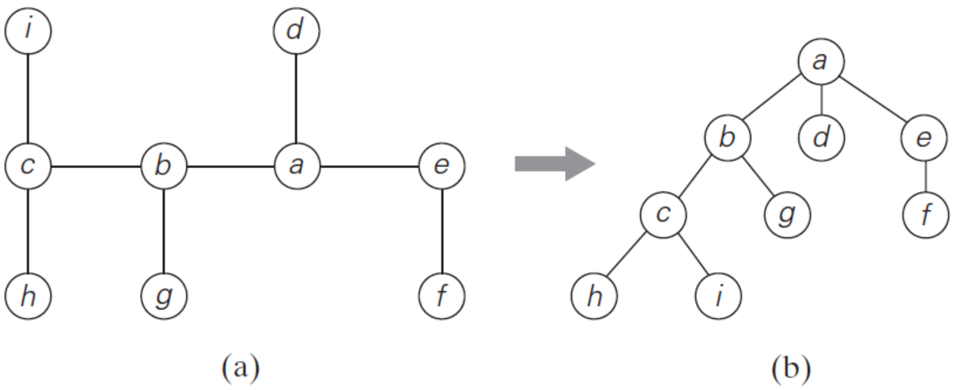


FIGURE 1.11 (a) Free tree. (b) Its transformation into a rooted tree.

Figure 4: rooted_tree

- **ancestor** of vertex v: all vertices on simple path from root to vertex v

  - vertex usually considered its own ancestor
  - **proper ancestor** excludes the vertex itself
- if

$$(u, v)$$

  is the last edge of simple path from root to vertex v

  - u is **parent** of v
  - v is **child** of u
- **sibling**: vertices with same parents
- **leaf**: vertex with no children
- **parental**: vertex with at least one child
- **descendants**: all vertices for which v is an ancestor
  - **proper descendants**: excludes v itself
- **subtree** rooted at v: all descendants of v with all edges connecting
- **depth** of a vertex v: length of simple path to v
- **height** of a tree: longest simple path from root to leaf

Ordered trees

- **ordered tree**: rooted tree in which all children of each vertex are ordered
- **binary tree**: ordered tree where each vertex has at most two children
  - each child is a **left child** or a **right child**
  - binary tree with root at left child of a vertex in a binary tree is the **left subtree**
  - as subtrees are also binary trees, they are useful for recursive algorithms
  - inequality for height *h* of a binary search tree with *n* nodes:

$$\lfloor \log_2 n \rfloor \leq h \leq n - 1$$

- **binary search tree**: numbers assigned to vertices, with parent vertex being larger than all elements in left subtree, and smaller than all elements in right subtree
- **multiway search tree**: generalisation of binary search trees
  - useful for efficient access to very large datasets
- **first child-next sibling** representation: left subtree of vertex is child, while right subtree is siblings.

- useful for computer representation of an arbitrary ordered tree with widely varying numbers of children by converting to a binary tree

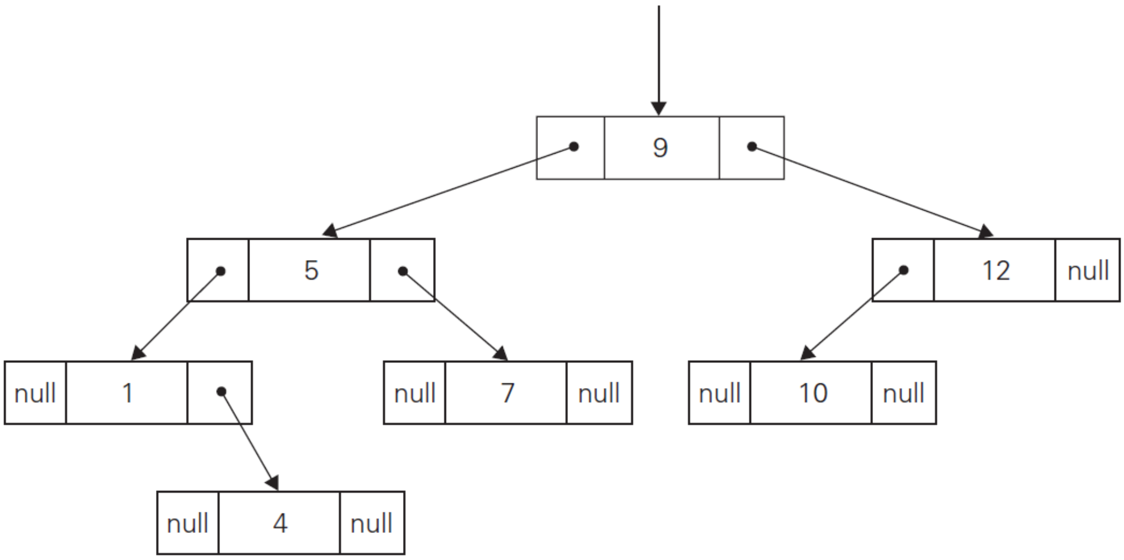


Figure 5: binary_search_tree

Sets and Dictionaries

- **set**: unordered collection of *distinct* **elements**
- operations:
  - checking membership
  - finding union
  - finding intersection

Universal set

- consider large set U with n elements
- **bit vector**: subset S of U can be represented by bit string of size n

e.g.

$$U = \{1, 2, 3, 4, 5, 6, 7, 8\} S = \{2, 3, 7\}$$

- bit string: 01100010
- these set representations allow very fast set operations but with high memory use

List structure

- more common approach for handling sets
- **multiset/bag**: circumvents uniqueness set requirement with an unordered collection of items that are not necessarily distinct
- lists are ordered, where as sets are not: largely this doesn't matter for practical purposes

Dictionary

- **dictionary**: data structure that implements most common set operations:
  - searching for an item
  - adding items
  - deleting items
- many implementations, from arrays to hashing and balanced search trees